MATH 2028 Honours Advanced Calculus II
2021-22 Term 1
Problem Set 4
due on Oct 11, 2021 (Monday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Gradescope on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: Throughout this problem set, we use (r, θ), (r, θ, z) and (ρ, ϕ, θ) to denote the polar, cylindrical and spherical coordinates respectively.

Problems to hand in

1. Let $\Omega \subset \mathbb{R}^2$ be the region bounded below by $y = 1$ and above by $x^2 + y^2 = 4$. Evaluate
$$\int_\Omega (x^2 + y^2)^{-3/2} \, dA.$$
2. Evaluate the iterated integral
$$\int_0^1 \int_y^1 \frac{xe^x}{x^2 + y^2} \, dx dy.$$
3. Find the volume of the region lying above the plane $z = a$ and inside the sphere $x^2+y^2+z^2 = 4a^2$ by integrating in cylindrical coordinates and spherical coordinates.
4. Find the volume of the region in $\mathbb{R}^3$ bounded by the cylinders $x^2 + y^2 = 1$, $y^2 + z^2 = 1$, and $x^2 + z^2 = 1$.

Suggested Exercises

1. Find the area enclosed by the cardioid in $\mathbb{R}^2$ expressed in polar coordinates as $r = 1 + \cos\theta$.
2. Let $\Omega \subset \mathbb{R}^2$ be the annularregion bounded by $x^2 + y^2 = 1$ and above by $x^2 + y^2 = 2$. Evaluate $\int_\Omega y^2 \, dA$.
3. Find the volume of the region in $\mathbb{R}^3$ bounded above by $z = 2$ and below by $z = x^2 + y^2$.
4. Find the volume of the region $\mathbb{R}^3$ inside both $x^2 + y^2 = 1$ and $x^2 + y^2 + z^2 = 2$.
5. Find the volume of a right circular cone of base radius a and height h by integrating in cylindrical coordinates and spherical coordinates.
6. Let $\Omega \subset \mathbb{R}^3$ be the region bounded below by the sphere $x^2+y^2+z^2 = 2z$ and above by the sphere $x^2 + y^2 + z^2 = 1$. Evaluate the integral
$$\int_\Omega \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \, dV.$$

7. (a) Let $\epsilon > 0$ be fixed. Show that there is a C^∞ function $g : \mathbb{R} \to [0, 1]$ such that $g(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x \geq \epsilon$.
   (b) Let $\Omega \subset \mathbb{R}^n$ be an open set and $K \subset \Omega$ be a compact subset. Prove that there exists a C^∞ function $f : \Omega \to [0, 1]$ such that $f(x) = 1$ for all $x \in K$.

Challenging Exercises

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded subset with measure zero $\partial\Omega$. Show that for any $\epsilon > 0$, there exists a compact subset $K \subset \Omega$ such that ∂K has measure zero and $\mathrm{Vol}(\Omega \setminus K) < \epsilon$.

2. (a) Let $S \subset \mathbb{R}^n$ be an arbitrary subset and $x_0 \in S$. We say that a function $f : S \to \mathbb{R}$ is differentiable at x_0 of class C^1 if there exists a C^1 function $g : U \to \mathbb{R}$ defined in a neighborhood U of x_0 in $\mathbb{R}^n$ such that $g = f$ on $U \cap S$. Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function whose support lies in U. Show that the function $h : \mathbb{R}^n \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \varphi(x)g(x) & \text{when } x \in U \\ 0 & \text{when } x \notin \mathrm{spt}(\varphi) \end{cases}$$

   is a well-defined C^1 function on $\mathbb{R}^n$.
   (b) Prove the following statement: if $f : S \to \mathbb{R}$ is differentiable of class C^1 at each $x_0 \in S$, then f may be extended to a C^1 function $h : \Omega \to \mathbb{R}$ defined on an open subset Ω containing S.